Hysteresis in Freeway Travel Time Variability

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SHORT SUMMARY

This study examines the relationship between mean and variance of travel times on a congested corridor using LWR theory, focusing on a freeway with a bottleneck and stochastic peak demand. We establish conditions for typical counterclockwise hysteresis loops and explain why deviations remain limited. Supported by numerical experiments, our results enhance understanding of hysteresis patterns and aid in traffic planning and control.

Keywords: traffic flow theory, hysteresis, traffic management.

1 INTRODUCTION

Travel time reliability is a critical aspect of traveler route choice in urban areas, with empirical analyses showing it is nearly as important to travelers as the expected travel time itself (e.g. (Prato, Rasmussen, & Nielsen, 2014)). Travel time variability provides a formal interpretation for reliability, which we define as the relationship between mean and variance of travel time, is often approximated by a linear curve (e.g. (Kim & Mahmasani, 2015)). However, empirical data show an anti-clockwise hysteresis loop between these two quantities, both at the level of individual links ((Fosgerau, 2010), (Kim & Mahmasani, 2015), (Yildirimoglu, Limniati, & Geroliminis, 2015)) and at a network level ((Gayah, Dixit, & Guler, 2015)). The aim of this work is to investigate the time-dependent relationship between mean and variance of travel time in a single corridor in the context of rush hour specific traffic dynamics. We analytically derive conditions for the occurrence of anti-clockwise hysteresis loops and explain which traffic flow variables determine the shape and size of the hysteresis loop. Fosgerau (Fosgerau, 2010) theoretically proves the occurrence of such loops in a queuing system with decreasing arrival rate. Yildirimoglu et al. (Yildirimoglu et al., 2015) use attribute the hysteresis in the day-to-day travel time variability to stochastic parameters of vehicle travel time. Separate from the hysteresis in travel time variability are other phenomena in traffic research that bear the same name: Treiterer and Myers (Treiterer & Myers, 1974) define hysteresis as the separation of speed-density curves into an accelerating and a decelerating branch ahead of traffic disturbances. Zhang (Zhang, 1999) and Yeo and Skabradonis (Yeo & Skabardonis, 2009) offer theoretical explanations for this effect. At network level, clockwise hysteresis loops caused by rush hour congestion have been demonstrated in networks with particular topologies (e.g. (Buisson & Ladier, 2009), (Saberi & Mahmassani, 2012)). This is due to differences in the spatial distribution of vehicles during congestion on- and offset (Geroliminis & Sun, 2011) and the increased network instability during the offset of congestion (Gayah & Daganzo, 2011).

2 METHODOLOGY

We model traffic flow on a link of length l with a downstream bottleneck capacity q_{bn} using the LWR ((Lighthill & Whitham, 1955), (Richards, 1956)) theory. The upstream boundary flow $q_{\mathrm{up}}(t)$ follows a trapezoidal, piece-wise linear function governed by a probability distribution ϕ. The space-time trajectory of the queue's tail is $\psi(t)$. The travel time τ for a vehicle entering the segment at time t is given by the following expression:

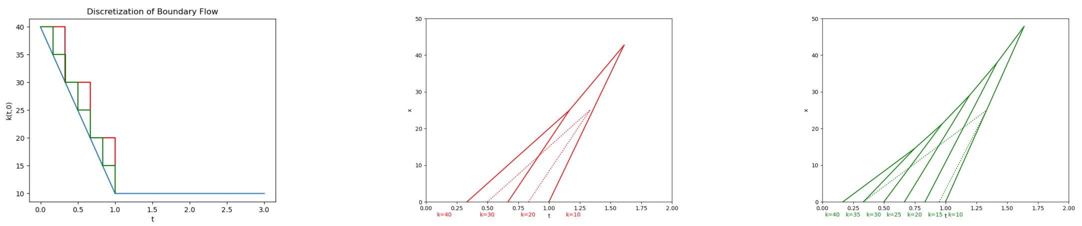
$$\tau(t) = \inf\{T \geq 0 : N(l, t+T) > N(0, t)\}, \tag{1}$$

We assume the queue length does not exceed the road segment's capacity.Then, the following general solution can be formulated:

Lemma 2.1. *For every point (x, t) satisfying $x < \psi(t)$ which is reached by at least one characteristic curve, the physically correct characteristic is the latest emanating one.*

Proof (Sketch). The lemma is proven by analyzing a discretized approximation of the upstream boundary condition. Assuming the concavity of $q(k)$, characteristics may intersect only if originating from the descending part of the boundary. We partition the decreasing branch into intervals $I_1, \ldots, I_n$. Define $q_{\text{discr}}(t)$ as $q_{\text{up}}(t_I)$ where t_I is the lower bound of the interval I containing t. Additionally, we linearize $q(k)$ over the decreasing branch. Suppose two characteristic lines intersect at (x, t), with c_1 from I_1 and c_2 from I_2, $I_1 < I_2$, and c_2 is most recent. c_1 must have crossed a shockwave, representing the physically valid solution at this point in space-time. The continuous boundary condition solution derives from this discretization method as intervals approach zero length. □

Figure 1 illustrates the lemma's approximation method. Figure 1a displays the transformation of a continuously decreasing boundary flow (blue) into three (red) or six (green) discrete steps. Figures 1b and 1c show the resulting solutions. These steps propagate as shock waves. The characteristics that intersect the point $(\frac{4}{3}, 25)$ are shown as dotted lines. It is straightforward to verify graphically that only the later emanating characteristic represents a feasible solution of the LWR theory in both cases.



(a) Discretized flow (b) Approximate solution 1 (c) Approximate solution 2

Figure 1: Visualization of the discretized flow and approximate solutions

3 RESULTS AND DISCUSSION

Lemma 3.1. *Let q_p be a random variable, and let τ_1 and τ_2 be functions of q_p such that $\mathbb{E}[\tau_1] = \mathbb{E}[\tau_2]$ and $\Delta\tau := \tau_2 - \tau_1$ is a convex function for $k \leq k_c$. Also, assume that $\tau_1(q_p) > 0$ and τ_1 is an increasing function of q_p. Then, it holds that*

$$\text{Var}[\tau_2] \geq \text{Var}[\tau_1].$$

Proof.

$$\text{Var}(\tau_2) = \text{Var}(\tau_1 + \Delta\tau) = \text{Var}(\tau_1) + \text{Var}(\Delta\tau) + 2\,\text{Cov}(\tau_1, \Delta\tau).$$

Expanding the covariance term:

$$\text{Cov}(\tau_1, \Delta\tau) = \mathbb{E}[\tau_1 \Delta\tau] - \mathbb{E}[\tau_1]\mathbb{E}[\Delta\tau],$$

and substituting, we have:

$$\text{Var}(\tau_2) = \text{Var}(\tau_1) + \text{Var}(\Delta\tau) + 2(\mathbb{E}[\tau_1 \Delta\tau] - \mathbb{E}[\tau_1]\mathbb{E}[\Delta\tau]).$$

Given that $\mathbb{E}[\tau_1 \Delta\tau] = \int \tau_1(\Delta\tau\psi(q))\, dq_p$, and considering that $\Delta\tau\psi(q)$ has a unique zero at q_0 such that $\Delta\tau(q)\psi(q) < 0$ for $q < q_0$ and $\Delta\tau(q)\psi(q) > 0$ for $q > q_0$, and the integral over this product is zero, it follows that: $\mathbb{E}[\tau_1 \Delta\tau] \geq 0$. The statement of the lemma follows. □

The proposed clockwise movement of travel time in the $(\mathbb{E}, \text{Var})$ plane is then a corollary of the following proposition:

Proposition 3.1. *Assume that $\tau_1 = \tau(t_1)$, $\tau_2 = \tau(t_2)$ with $t_1 \leq t_2$, that $\mathbb{E}[\tau_1] = \mathbb{E}[\tau_2]$ and that $v(k) = q(k)/k$ is a convex function. Then,* $\text{Var}[\tau_2] \geq \text{Var}[\tau_1]$.

Proof. We show that $\Delta\tau$ is convex under the given assumptions. We define $\tau_f(t)$ as the travel time incurred by a vehicle starting at t if there were no standing queue, i.e. in the case of $q_{bn} = \infty$ and $\tau_q(t)$ as the residual travel time caused by queuing: $\tau(t) = \tau_f(t) + \tau_q(t)$.

$\Delta\tau_f$: Omitted.

$\Delta\tau_q$: Let $\hat{q_{p,1}}$ be the earliest time so that the vehicle starting at time t_1 experiences queuing and let $\hat{q_{p,2}}$ be the analogously defined time for t_2. Since $\Delta\tau_q(t) = 0$ applies for $q_p \leq \hat{q_{p,1}}$, $\Delta\tau_q(t)$ is convex in this interval. For $\hat{q_{p,1}} \leq q_p \leq \hat{q_{p,2}}$, $t_q(t2) = 0$ and $\frac{d}{dq_p}\tau_q(t_1) = \frac{d}{dq_p}N_0(t_1) \cdot \frac{1}{q_{bn}} = \frac{\frac{d}{dq_p}\int_0^{t_1} q(0,t)\,dt}{q_{bn}}$, which is a constant in q_p due to the trapezoidal nature of the upstream boundary flow. We can conclude that $\Delta\tau_q(t)$ is convex in this interval. For all $t \geq \hat{\tau}_2(q_p)$, the term $\Delta\tau_q$ increases according to the time necessary for the queuing capacity to process vehicles that start at the upstream boundary between times t_1 and t_2, i.e. $\frac{d}{dq_p}\Delta\tau_q(t) = \frac{d}{dq_p}\left(\int_{t_1}^{t_2} q(0,t)\,dt\right) \cdot \frac{1}{q_{bn}}$. Given the trapezoidal shape of the boundary flow, it follows that $\Delta\tau_q$ is a linearly increasing function over the interval $[t_1, t_2]$. Combining these observations, $\Delta\tau_q$ exhibits piecewise linear behavior, decreasing initially and then increasing, thereby rendering it convex over its entire domain.

□

The simulation scenarios using the Cell Transmission Model (CTM) illustrate the theorized hysteresis dynamics . Simulations span 240 time units on a 40-unit corridor with a downstream bottleneck capacity of 25 vehicles per time unit. The fundamental diagram is triangular with a free-flow speed of 1, a critical density k_c of 60, and a jam density k_j of 240.

The upstream boundary flow is given by $q(0,t) = 20 + \frac{(q_{\max}-20)}{60} \times t$ for $t \in [0,60]$, $q_{\max}$ for $t \in [60,90]$, $10 + q_{\max} - \frac{q_{\max}}{60} \times (t-90)$ for $t \in [90,150]$, 10 for $t \in [150,180]$, and 0 for $t \in [180,240]$ The value of $q_{\max}$ is normally distributed. 300 simulations were executed across high ($\mu = 40$) and low ($\mu = 30$) demand scenarios with standard deviations of 10, 15, 20 and 25 of the mean, respectively. Hysteresis is quantified by the area within the loop.

Figure 2: Travel Time And Variances, Low Variance of Boundary Demand

Figure 3: Travel Time And Variances, High Variance of Boundary Demand

Linear regression models for fixed means exhibit excellent fits with slopes $m_{30} \approx 114.79$ and $m_{40} \approx 1088.29$, and coefficients of determination $R^2 = 0.9962$ and $R^2 = 0.9970$. These results are supported by Proposition 3.1, highlighting that excess variance between start times t_1 and t_2 reflects sensitivity to changes in q_p rather than variance of $\phi(q_p)$, as shown by the piecewise linear form of $\Delta\tau_q$. For the triangular shape of the fundamental diagram, stochasticity of demand has no effect on travel times in uncongested conditions. Graphically, increases in σ do not affect the horizontal mean distance, but mainly increase the vertical variance. Additionally, our analyses indicate that hysteresis magnitude is more influenced by changes in mean peak demand than by its variance.

4 CONCLUSIONS

The presented model provides an explanation for the causes of the form and extent of hysteresis in the relationship between mean and variance of travel time. It allows to quantify this type of hysteresis without prior empirical validation, and to predict under which conditions a deviation from the known and expected counter-clockwise direction of travel is possible. Lemma 3.1 provides a powerful and physically meaningful characterization of systems which exhibit this type of hysteresis. Proposition 3.1 derives the convexity property for the delay caused by queuing effects: when the boundary flow increases, the upstream flow between t_1 and t_2 increases linearly, while the bottleneck capacity remains unchanged. For the delay caused by changes in the uncongested regime, however, a strictly concave speed-density relationship can reverse this effect. However, curves of this form are generally unusual in traffic flow modeling and only a minor role for travel time delay in real-life transportation networks, which explains why clockwise movements are only rarely observed empirically and, if at all, usually only as a sub-loop of a larger counter-clockwise movement.

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