Exact numerical solution method for a train eco-driving problem

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1 INTRODUCTION

Railway is an important mode of transport due to its high capacity, punctuality and sustainability. Nonetheless, the overall energy consumption of railway systems is substantial worldwide as the railway networks are rapidly expanding. More than 50% of the energy on railway operation is consumed by train traction systems. Eco-driving is widely regarded as one of the effective measures to reduce train operation energy, as the energy consumption for train traction is mainly determined by train driving speed profiles. The aim of eco-driving is to find the most energy-efficient driving style that can satisfy realistic operational constraints and adhere to a predefined travel time between two stops.

The simplest train eco-driving problem investigated a train running on a flat track with uniform speed limits. To comply with actual situations, more practical conditions and constraints, such as varying track gradients, varying speed limits, and nonlinear train characteristics, were considered (see, e.g., Scheepmaker *et al.* (2017), for a review). The classic train eco-driving problem is usually formulated as an optimal control problem (OCP). The OCP should be solved to obtain speed and control profiles that can be used to guide train operations. In the literature, there are two main kinds of methods to solve the OCP: indirect methods, direct methods.

Indirect methods, using Pontryagin's maximum principle, first derive the analytical properties of optimal control modes, and then design numerical algorithms to calculate the sequence of these modes and the positions of their changing over (Albrecht *et al.*, 2016). The complexity of the algorithm design depends on the complexity of the grade profile and speed limits, since each new problem requires tailored/sophisticated analysis on the optimality conditions. Different from the indirect methods that need to analyse the optimality conditions of the OCP, by discretizing the independent variable or state variables, the direct methods convert the OCP into a nonconvex nonlinear program (NLP) or a graph formulation. Various techniques used to solve the problems can be further summarized as three categories: 1) the first category of methods directly solves the nonconvex NLP, such as pseudospectral method (Wang & Goverde, 2016, Ye & Liu, 2016); 2) the second category of methods converts the nonconvex NLP (via approximation or relaxation) into other forms that can be solved to global optimum, such as approximation to mixed-integer linear programs (MILPs) (Wei *et al.*, 2022) or convex programs (CPs) (Xiao *et al.*, 2023), and relaxation to CPs (Ying *et al.*, 2023); 3) the third category of methods discretizes both of the independent variable and the state variables to construct graph formulations, such as space-speed discretized graph (Haahr *et al.*, 2017) and space-time-speed discretized graph (Wang *et al.*, 2021), which can be solved by (tailored) dynamic programming. But so far these methods either cannot solve to global optimum or consume very long computing time.

This paper focuses on developing a new direct method that can solve the classic train eco-driving problem to global optimum. In comparison with the existing methods, the main contributions of this paper are highlighted as follows: 1) For the classic train eco-driving problem, we reformulate the nonconvex constraints as either convex or bilinear constraints, and the reformulated model has the same optimal solutions as the original model. The reformulated model can be efficiently solved to globally optimal solutions using off-the-shelf solvers. 2) Extensive numerical experiments are conducted to evaluate the performances of our proposed methods in terms of solution quality and computation efficiency.

2 Methodology

We first present a widely-used formulation for modeling the classic single-train eco-driving problem which uses train location s as independent variable kinetic energy per unit of mass $E(s)$, i.e., $E = \frac{v^2}{2}$, and travel time $t(s)$ at location s as state variables (Wei *et al.*, 2022, Xiao *et al.*, 2023). The goal of the train eco-driving problem is to drive a train from a given position s_0 to a given position s_f within a predefined trip time T, while minimizing net energy consumption. The eco-driving problem can then be formulated as:

$$\min \int_{s_0}^{s_\mathrm{f}} F^+(s)\mathrm{d}s \tag{1a}$$

$$\text{s.t.} \frac{\mathrm{d}E(s)}{\mathrm{d}s} = \frac{F(s) - 2\mathrm{c}_2 E(s) - \mathrm{c}_1\sqrt{2E(s)} - \mathrm{c}_0 - mg\sin(\alpha(s))}{m} \tag{1b}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{\sqrt{2E(s)}} \tag{1c}$$

$$\epsilon_\mathrm{v}^2/2 \leq E(s) \leq v_\mathrm{max}^2(s)/2 \tag{1d}$$

$$F_\mathrm{min} \leq F(s) \leq F_\mathrm{max} \tag{1e}$$

$$P_\mathrm{min} \leq F(s)\sqrt{2E(s)} \leq P_\mathrm{max} \tag{1f}$$

$$E(s_0) = E(s_\mathrm{f}) = \epsilon_\mathrm{v}^2/2. \tag{1g}$$

$$t(s_\mathrm{f}) - t(s_0) \leq T \tag{1h}$$

$$F^+(s) \geq F(s) \tag{1i}$$

$$F^+(s) \geq \eta_\mathrm{reg} F(s) \tag{1j}$$

where m is the total mass including static mass and rotating mass; $F(s)$ is the force applied at wheels at position s, positive for traction and negative for braking; c_0, c_1 and c_2 are positive coefficients of the running resistance; g is the gravitational acceleration and $\alpha(s)$ is track gradient at location s; $\eta_\mathrm{reg} \in [0, 1)$ denotes the proportion of braking energy that can be reused; and $F^+(s)$ represents the positive value of the force applied at wheels at position s.

We present two steps to reformulate the model (1). The two steps are: kinetic dynamics reformulation and time dynamics convexification. After that, we can prove optimal solutions of the reformulated model are the same as the original model (1). We use both the kinetic energy E and speed v as optimization variables. Following the approach presented in Xiao *et al.* (2023), a new variable z is introduced to convexify (1c). Combining the new variable with (1c), we have

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = z(s) \tag{2a}$$

$$z(s) \geq \frac{1}{v(s)}. \tag{2b}$$

Towards the resolution of the continuous model, we recast it into an NLP by discretization. The journey between two stations is split into N intervals by choosing a set of discrete points s_k, with $s_0 = s_0$ and $s_N = s_\mathrm{f}$, and $\Delta s_k = s_k - s_{k-1}$ for $k = 1, ..., N$. We choose a piecewise constant control parametrization, i.e., $F(s) = F(k)$, $s \in [s_{k-1}, s_k)$. The NLP is summarized as:

$$\min.\sum_{k=1}^{N} F^{+}(k)\Delta s_k \tag{3a}$$

$$\text{s.t.}\ \frac{E(k)-E(k-1)}{\Delta s_k} = \frac{F(k) - 2\mathrm{c}_2 E(k) - \mathrm{c}_1 v(k) - \mathrm{c}_0 - mg\sin(\alpha(k))}{m} \tag{3b}$$

$$\frac{t(k)-t(k-1)}{\Delta s_k} = z(k) \tag{3c}$$

$$z(k) \geq \frac{1}{v(k)} \tag{3d}$$

$$E(k) = \frac{v^2(k)}{2} \tag{3e}$$

$$F_\mathrm{min} \leq F(k) \leq F_\mathrm{max} \tag{3f}$$

$$P_\mathrm{min} \leq F(k)v(k) \leq P_\mathrm{max} \tag{3g}$$

$$\epsilon_\mathrm{v}^2/2 \leq E(k) \leq v_\mathrm{max}^2(k)/2 \tag{3h}$$

$$\epsilon_\mathrm{v} \leq v(k) \leq v_\mathrm{max}(k) \tag{3i}$$

$$E(0) = E(N) = \epsilon_\mathrm{v}^2/2 \tag{3j}$$

$$v(0) = v(N) = \epsilon_\mathrm{v} \tag{3k}$$

$$t(N) - t(0) \leq T \tag{3l}$$

$$F^{+}(k) \geq F(k) \tag{3m}$$

$$F^{+}(k) \geq \eta_\mathrm{reg} F(k). \tag{3n}$$

In the model (3), the cost function is linear, and the constraints are biliner or convex, which can be solved to global optimum by off-the-shelf solvers such as Gurobi.

3 RESULTS

This section presents numerical results of the model (3) on an artificial but practical trip. The planned trip time is 600 s. The gradients and speed limits are given in Fig. 1. First, we verify the exactness of the convex relaxation (3d). The optimal solutions for $\eta_\mathrm{reg} = 0$ and $\eta_\mathrm{reg} = 0.5$ are given in Fig. 1. We can observe that speed profiles are below the speed limits, control force profiles are within the force bounds, and the relaxation (3d) is always tight. The tightness of the relaxation verifies that optimal solutions of the classic eco-driving problem can be generated by solving the proposed model (3).

Second, we evaluate the advantages of the proposed convex relaxation. We solve a bilinear model that replaces the convex relaxation (3d) with a bilinear equality, i.e., $z(k)v(k) = 1$, which can be solved to global optimum by Gurobi 11.0 in theory. The computing time limit is set to 1800 seconds. The numerical results are summarised in Table 1. For most instances of the bilinear model, global optimum is not achieved within 1800 seconds. However, all instances of the proposed model are solved to global optimum within one minute.

By convexifying the time dynamics, the nonconvex time constraints of the eco-driving problems can be efficiently solved. Mathematical proofs of the lossless convexification can be proposed (in our future paper) and their validity is demonstrated by extensive numerical experiments, which indicates that our proposed methods can deliver exact numerical solutions for the eco-driving problem. Furthermore, the proposed method can be extended to consider the situation with interference on trains, such as time window constraints.

Table 1 – *Performances of the proposed method.*

Instance	The proposed model (3)		The bilinear model	
	Cost value [kWh]	Compt. time [s]	Compt. time [s]	Optimality gap [%]
$\eta_{\text{reg}} = 0$	172.033	16.514	22.869	0
$\eta_{\text{reg}} = 0.2$	167.999	24.266	1800.111	0.887
$\eta_{\text{reg}} = 0.4$	163.820	42.492	1800.133	0.740
$\eta_{\text{reg}} = 0.6$	159.374	26.162	1801.017	0.761
$\eta_{\text{reg}} = 0.8$	154.425	40.634	1801.425	0.622

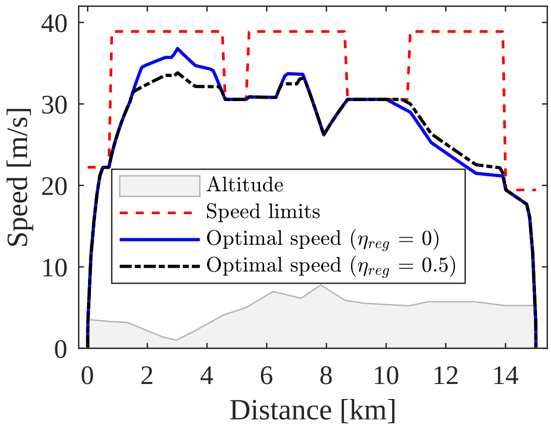

Figure 1 – *Optimal speed profiles of the proposed method.*

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